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Estimation of System Matrices by Dynamic Condensation and Application to Structural Modification

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A method for estimating mass, stiffness, and damping matrices of a dynamic system was presented in our previous paper. That technique is based on extracting the normal frequency response functions from the experimental complex ones. Independent from the mass and stiffness matrices, the damping matrix is calculated alone, whereas the mass and the stiffness matrices are identified from the normal frequency response functions by using the least squares method. In this paper, we extend that method to deal with an incomplete system and to indicate that the condensed physical system matrices can also be obtained by that method. These condensed system matrices represent a dynamic condensation of the original full system. Furthermore, the condensed system matrices are applied to the problem of determining the structural modifications necessary to shift the damped natural frequency to a desired value. Both simulation and experimental examples are carried out to illustrate the applicability of the present method. The results indicate that the method is effective and accurate.

Nomenclature

C = viscous damping matrix of original system
C_r = viscous damping matrix of reduced system
$\bar{f}(t)$ = force vector of original system in time domain
$f(\omega)$ = force vector of original system in frequency domain
$f_r(\omega)$ = force vector of reduced system in frequency domain
$H^C(\omega)$ = complex frequency response function matrix for the original system
$H_r^C(\omega)$ = complex frequency response function matrix for the reduced system
$H^N(\omega)$ = normal frequency response function matrix for the original system
$H_r^N(\omega)$ = normal frequency response function matrix for the reduced system
$H(\lambda)$ = transfer function matrix in Laplace domain
I = identity matrix
K = stiffness matrix of original system
K_r = stiffness matrix of reduced system
M = mass matrix of original system
M_r = mass matrix of reduced system
$\bar{x}^C(t)$ = response vector in time domain for the original damped system
$x^C(\omega)$ = response vector in frequency domain for the original damped system
$x_r^C(\omega)$ = response vector in frequency domain for the reduced damped system
$x^C(\lambda)$ = response vector in Laplace domain for damped system
ΔC = change of damping matrix C
ΔK = change of stiffness matrix K
ΔK_p = matrix ΔK obtained from changing only one stiffness k_p
ΔM = change of mass matrix M
ΔM_Q = matrix ΔM obtained from changing only one lumped mass m_Q
δk_p = change of stiffness k_p
δm_Q = change of lumped mass m_Q at node Q
λ = Laplace variable, i.e., $\lambda = -\sigma + i\omega$
ω = frequency of excitation
$[\ ]^{-1}$ = inverse of a matrix
$[\ ]^T$ = transpose of a matrix
$[\ ]_I$ = imaginary part of a complex matrix
$[\ ]_R$ = real part of a complex matrix

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I. Introduction

BECAUSE of the increasing complexity of modern mechanical structures, an accurate mathematical model [such as a finite element (FE) model] has become a necessity for successful design of mechanical systems. We often have the problem that a mathematical model does not predict the dynamic behavior of the real system well when compared with the measurements. The discrepancy between model response and real measurements may be due to inaccurate parameters of the model. Hence, more accurate matrices for mass, stiffness, and damping of a structure model are needed for developing better active vibration control system, for predicting accurate responses, for estimating loads, and for making structural modification. One realizable approach is to tune the FE model using the measurement data. However, since the degrees of freedom (DOF) of a FE model are much larger than those of measurements, the model must be reduced to match the DOF of measurement. Most reduction techniques will alter the dynamic characteristics contained in the original full analytical model. Generally, the estimated frequencies in the reduced model are higher than those of the original model. Guyan[1] reduction, commonly referred to as static condensation, is probably the most widely used condensation technique for the reduction of large analytical models. The amount of error introduced in the Guyan reduction process, however, is heavily dependent on the selection of the degrees of freedom to be retained. O'Callahan[2] proposed the improved reduced system (IRS) technique to improve the results obtained from Guyan reduction by taking into account knowledge of system inertial effects. Paz[3] proposed an algorithm for dynamic condensation. That algorithm is applied progressively from the fundamental mode to any desired number of higher modes, and an iterative process may be implemented to further improve the solution. These methods are all based on an analytical model such as the FE model.

Alternatively, many researchers focus their efforts on the estimation of system matrices directly from the measurement data of structures. In the modeling of a structure, the damping matrix is more difficult to identify than that of the mass and the stiffness matrices from noisy measurement data. Several methods[4–8] have been developed for estimating the mass, the stiffness, and the damping matrices of the mathematical model of a structure. Observing the results of these works, one may find that the estimated damping matrix has larger relative error than that of the mass and the stiffness matrices. The reason is that the order of the damping coefficients is often much smaller than that of the stiffness coefficients. For small coefficients, larger relative error will occur when the least squares method or instrument variable method is employed to calculate these coefficients, namely, stiffness, mass and damping, together. The accuracy will be improved if the estimation of the damping matrix can be separated from the estimation of mass and stiffness matrices.

In previous work,[9] a relationship between the complex and the normal frequency response functions (FRFs) was formulated. By using this relationship, a frequency-domain technique for estimating mass, stiffness, and damping matrices was also developed. In Ref. 9 the estimation of the damping matrix is separated from the estimation of mass and stiffness matrices. The normal FRFs extracted from the complex FRFs are employed to find the mass and stiffness matrices for a complete system. In this paper, we also apply the method to the incomplete system to obtain the condensed normal FRFs and the condensed physical system matrices. These condensed physical system matrices are then applied to dynamic structural modification for shifting the damped natural frequency to a desired value by changing the mass or stiffness of the original system. The method is practical because measured FRF data are used directly to achieve the goal. There is no need to construct a finite element model.

II. Identification of Physical System Matrices

For a structure with viscous damping, the equations of motion can be written as

$$M\ddot{\bar{x}}^C(t) + C\dot{\bar{x}}^C(t) + K\bar{x}^C(t) = \bar{f}(t) \tag{1}$$

where M, C, and K are the $n \times n$ mass, viscous damping, and stiffness matrices, respectively, and they are referred to as the physical system matrices of a structure. Since most estimations involve the calculation of mass, stiffness, and damping matrices simultaneously, large error often appears for estimating damping matrix. To reduce the error, our goal is to find a method to estimate the damping matrix independently. This can be achieved by the previous work[9] and is summarized as follows. The reader is referred to Ref. 9 for more detailed derivations.

For harmonic excitation, Eq. (1) can be expressed as

$$[H^N(\omega)]^{-1}\boldsymbol{x}^C(\omega) + i\omega C\boldsymbol{x}^C(\omega) = \boldsymbol{f}(\omega) \tag{2}$$

where $\bar{f}(t) = \boldsymbol{f}(\omega)e^{i\omega t}$, $\bar{x}(t) = \boldsymbol{x}(\omega)e^{i\omega t}$, and $H^N(\omega) = (-\omega^2 M + K)^{-1}$ are used. $H^N(\omega)$ is the normal FRF matrix. Premultiplying Eq. (2) by $H^N(\omega)$ yields

$$\boldsymbol{x}^C(\omega) + iG(\omega)\boldsymbol{x}^C(\omega) = H^N(\omega)\boldsymbol{f}(\omega) \tag{3}$$

where

$$G(\omega) = \omega H^N(\omega)C \tag{4}$$

is a real matrix, and it is referred to as the transformation matrix. Equation (4) provides a relationship between the transformation matrix G and the damping matrix C. If the transformation matrix and the normal FRF matrix H^N can be calculated, then the damping matrix can be identified independently. By rearranging Eq. (3) and using the complex response equation $\boldsymbol{x}^C(\omega) = H^C(\omega)\boldsymbol{f}(\omega)$, the transformation matrix $G(\omega)$ then can be solved in terms of the matrices $H_R^C(\omega)$ and $H_I^C(\omega)$ by

$$G(\omega) = -H_I^C(\omega)\left[H_R^C(\omega)\right]^{-1} \tag{5}$$

and the relation between the normal FRF matrix and the complex FRF matrix as

$$\begin{aligned} H^N(\omega) &= H_R^C(\omega) - G(\omega)H_I^C(\omega) \\ &= H_R^C(\omega) + H_I^C(\omega)\left[H_R^C(\omega)\right]^{-1}H_I^C(\omega) \end{aligned} \tag{6}$$

From Eqs. (5) and (6), the transformation matrix $G(\omega)$ and the normal FRF matrix $H^N(\omega)$ can be calculated, respectively, given $H^C(\omega)$. Once matrices $G(\omega)$ and $H^N(\omega)$ become available, the damping matrix can be calculated from Eq. (4).

Estimation of Damping Matrix

For the noise-free case, the exact solution for damping matrix can be obtained directly from Eq. (4) by

$$C = 1/\omega_j\left[H^N(\omega_j)\right]^{-1}G(\omega_j) \tag{7}$$

where ω_j is a chosen frequency. In practice, the frequency response functions are contaminated with noise and the least squares method (LS) is employed to solve for the damping matrix. By evaluating Eq. (4) at several frequencies $\omega_1, \omega_2, \ldots, \omega_m$, we have

$$VC = Q \tag{8}$$

where

$$V = \left[\omega_1 H^N(\omega_1) \quad \omega_2 H^N(\omega_2) \quad \ldots \quad \omega_m H^N(\omega_m)\right]^T$$

$$Q = [G(\omega_1) \quad G(\omega_2) \quad \ldots \quad G(\omega_m)]^T$$

Since C is a symmetric matrix, we define a parameter vector $\bar{c}$ from the lower triangular matrix of C:

$$\bar{c} = [c_{11} \quad c_{21} \quad c_{22} \quad c_{31} \ldots c_{ij} \ldots c_{nn}]^T \qquad i \geq j \tag{9}$$

where c_{ij} is the (i, j) element of damping matrix C. By using the outer product expansion and rearranging the V and Q matrices correspondingly, the real overdetermined equation can be rearranged as

$$\bar{V}\bar{c} = \bar{q} \tag{10}$$

where $\bar{V}$ and $\bar{q}$ are formed from the V and Q matrices, respectively. Note that the dimension of $\bar{V}$ is $mn^2 \times n(n+1)/2$ and that of $\bar{q}$ is $mn^2 \times 1$. The dimension of $\bar{V}$ and $\bar{q}$ can be further reduced by utilizing known zero elements of the C matrix. Equation (10) can be solved by the least squares method when $m \geq n$. This leads to the normal equation as

$$\bar{V}^T\bar{V}\bar{c} = \bar{V}^T\bar{q} \tag{11}$$

The solution vector $\bar{c}$ provides the required damping matrix C. In this work the damping matrix is identified independently from the mass and the stiffness matrices. This is the main difference from other existing methods. Next, the identification of mass and stiffness matrices is presented.

Estimation of Mass and Stiffness Matrices

For an undamped system, the equations of motion can be written as

$$(-\omega^2 M + K)\boldsymbol{x}^N(\omega) = \boldsymbol{f}(\omega) \tag{12}$$

The relation between the matrices M, K, and $H^N(\omega)$ is given by

$$(-\omega^2 M + K)H^N(\omega) = I \tag{13}$$

where $H^N(\omega)$ is calculated from Eq. (6). By transposing Eq. (13) and taking the symmetry of M, K, and $H^N(\omega)$ into account, we obtain

$$[A \quad B]\begin{bmatrix} M \\ K \end{bmatrix} = E \tag{14}$$

where

$$A = -\left[\omega_1^2 H^N(\omega_1) \quad \omega_2^2 H^N(\omega_2) \quad \ldots \quad \omega_m^2 H^N(\omega_m)\right]^T$$

$$B = \left[H^N(\omega_1) \quad H^N(\omega_2) \quad \ldots \quad H^N(\omega_m)\right]^T$$

$$E = [I \quad I \quad \ldots \quad I]^T$$

Similar to the derivation in the preceding section, the real overdetermined equation can also be obtained and solved by the least squares method.

Selection of Data Points on the Identification Procedure

In general, the data points near resonant peaks of the FRFs contain more information of the characteristics of a dynamic system and the signal to noise ratio is higher. From these points of view, it is preferred to select the data points near each resonant peak for the identification procedure.

III. Condensed Physical System Matrices

In practice, a real structure has infinitely many of degrees of freedom. Only finite complex FRFs associated with finite measurement locations (or degrees of freedom), however, can be obtained using modal testing. It should be noted that there is no truncation error for each complex FRF in the measured frequency range. From this point of view, if only r master degrees of freedom are chosen, the complex response equation becomes

$$x_r^C(\omega) = H_r^C(\omega) f_r(\omega) \tag{15}$$

where $H_r^C(\omega)$ is the $r \times r$ reduced complex FRF matrix associated with the measured degrees of freedom x_r^C. Similar to Sec. II the condensed transformation matrix is given as follows:

$$G_r(\omega) = -\left[H_r^C(\omega)\right]_I \left[H_r^C(\omega)\right]_R^{-1} \tag{16}$$

and the condensed normal FRF becomes

$$H_r^N(\omega) = \left[H_r^C(\omega)\right]_R - G_r(\omega)\left[H_r^C(\omega)\right]_I \tag{17}$$

From Eqs. (16) and (17), the condensed transformation matrix $G_r(\omega)$ and the condensed normal FRF matrix $H_r^N(\omega)$ can be calculated, respectively. Once matrices $G_r(\omega)$ and $H_r^N(\omega)$ become available, similar to the derivation in preceding section, the condensed physical system matrices M_r, C_r, and K_r can be obtained by least squares method. In the next section, the identified condensed physical system matrices are used for the dynamic structural modification.

IV. Dynamic Structural Modification

In previous work,[10] a structural modification method for shifting natural freqencies of damped mechanical systems was developed. The transfer functions are in the Laplace domain, and an iteration procedure is required. In experiments, however, only frequency response functions in the frequency domain and finite numbers of data for each frequency response function can be obtained. Hence, that technique can not be applied directly to real structures. Assume the structure can be modified by adding mass directly to a nodal DOF or by adding stiffness only between two nodal DOF in the condensed system. The identified condensed system matrices M_r, C_r, and K_r obtained in the preceding section can be utilized such that the modification technique becomes available for experimental data. The modification technique is summarized in the Appendix. More detailed derivation and interpretation can be found in Ref. 10. In the next section, both simulation and experimental examples are employed to illustrate the applicability of the proposed method.

V. Illustrative Examples

In this section, three examples are presented. Examples 1 and 2 are simulation examples, but an actual test structure is used in example 3.

Example 1

Figure 1 shows a 2-DOF lumped-mass system. Although it is a simple model, it provides a clear concept for the dynamic condensation. In this example, the sampling frequency ($\Delta\omega$) is 0.1 Hz and three frequency points, namely, 2.0, 2.5, and 3.0 Hz are used for the identification procedure of the present method. The first case is to retain x_1 (master DOF) in the reduction process, and only one complex FRF associated with the master DOF x_1 is measured for the present method. For this reduction, the original 2-DOF system turns out to be a 1-DOF reduced system with a set of equivalent parameters. Table 1 shows the equivalent mass and stiffness obtained by the Guyan reduction, the IRS method, and the present method, respectively. The equivalent mass from Guyan reduction is the sum of m_1 and m_2, and the equivalent stiffness is just k_1 (static condensation). The IRS method is an improvement of Guyan reduction. The effect of the inertia of the deleted mass is included in the reduction process. The present method estimates the condensed system matrices from the complex (measured) FRFs directly. The effects of the deleted mass m_2, stiffness k_2, and damping c are compensated automatically in the process of reduction.

Table 1 Comparison of equivalent mass and stiffness for retaining the first DOF as master DOF, example 1

Methods \ Parameters	Equivalent mass, kg	Equivalent stiffness, N/m
Guyan reduction	3	1000
IRS method	5.17	1296
Present method	5.49	1373

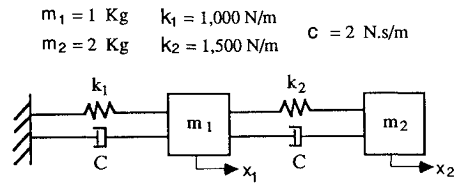

Fig. 1 Simulated system of 2 DOF.

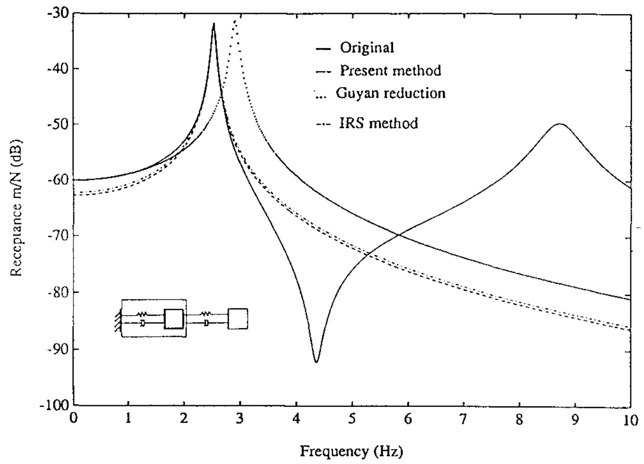

Fig. 2 Receptance FRF $h_{11}(\omega)$ obtained by selecting x_1 as master DOF for the 2-DOF system.

The corresponding receptance FRFs are shown in Fig. 2. It is obvious that the natural frequency obtained by Guyan reduction (dotted line) has larger deviations from the exact one; that is, the ratio of the equivalent mass and stiffness is not accurate. The natural frequencies obtained from IRS (dashed-dotted line) and from the present method (dashed line) are in good agreement with the exact one, i.e., the ratios of the two pairs of equivalent mass and stiffness are accurate. It should be noted, however, that for a real structure the IRS method can not be applied directly until an analytical model (such as the FE model) is established, whereas the present method can directly identify the condensed system matrices from the measured FRFs. This is the main difference between the present method and most of existing methods. The receptance FRF obtained by the three methods is shown in Fig. 3 for when x_2 of the 2-DOF system is chosen as master DOF. Note that the natural frequency from the IRS method is not accurate, although it is an improvement of Guyan reduction. The natural frequency obtained by the present method is again in good agreement with the exact one. Hence, the present method is not sensitive to the selection of the DOFs to be retained in the reduction process, but the Guyan reduction and the IRS method are heavily dependent on the selection of the DOFs.

Example 2

The second example is a 7-DOF, lumped-mass system as shown in Fig. 4. In this system, all of the lumped masses are selected to be unity. The modal parameters are listed in Table 2. The reason

Table 2 Eigenvalues of the original 7-DOF system

	Eigenvalue $\lambda_r = -\sigma_r + i\omega_r$	
Mode	Damping σ_r, rad/s	Damped natural frequency ω_r, rad/s
1	1.18	84.17
2	7.26	143.64
3	7.62	176.73
4	7.92	181.28
5	5.67	256.59
6	18.44	259.37
7	11.91	288.72

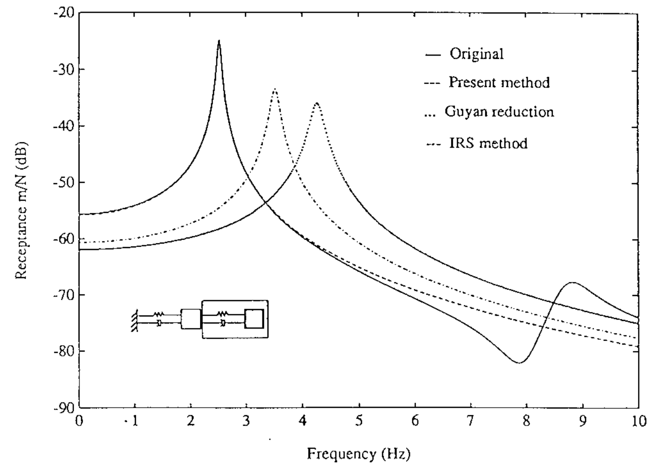

Fig. 3 Receptance FRF $h_{22}(\omega)$ obtained by selecting x_2 as master DOF for the 2-DOF system.

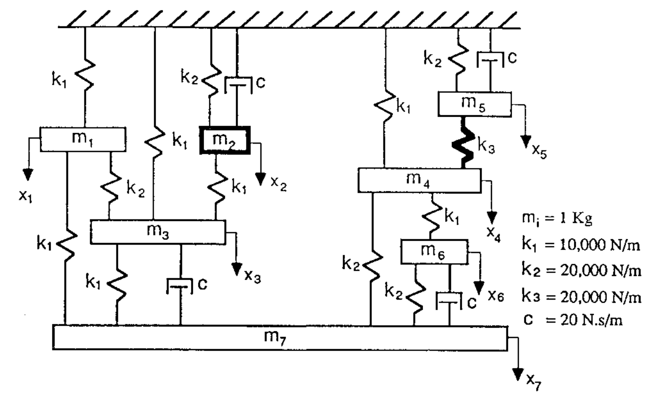

Fig. 4 Simulated system of 7 DOF.

the system was chosen is that there are two pairs of closely spaced modes, i.e., the third and the fourth mode (around 28 Hz) and the fifth and the sixth mode (around 41 Hz). The highest damping ratio is about 7% for the sixth mode. Hence, it is referred to as a highly coupled system. In this example, the sampling frequency ($\Delta\omega$) is 0.1 Hz and 11 frequency points, namely, 13.0, 13.5, 14.0, 22.0, 22.5, 23.0, 27.5, 28.0, 28.5, 29.0, and 29.5 Hz are used for the identification procedure of the present method. A typical FRF is shown in Fig. 5 (solid line). Only five peaks can be observed in the magnitude plot although there are seven modes for the system. Figure 5 also shows the results of reduction by each method with the master DOFs x_1, x_2, x_5, and x_6 (4 DOF). It is obvious that the IRS is an improvement of Guyan reduction, but some deviations for the natural frequencies are found. On the other hand, the first four natural frequencies obtained by the present method are in good agreement with the exact original first four natural frequencies. A comparison of the results of example 1 and example 2 shows that all stated conclusions of example 1 are also valid for example 2.

Next, the condensed system matrices identified by the present method are applied to the problem of determining appropriate structural modification. Figure 6 shows the relation between the

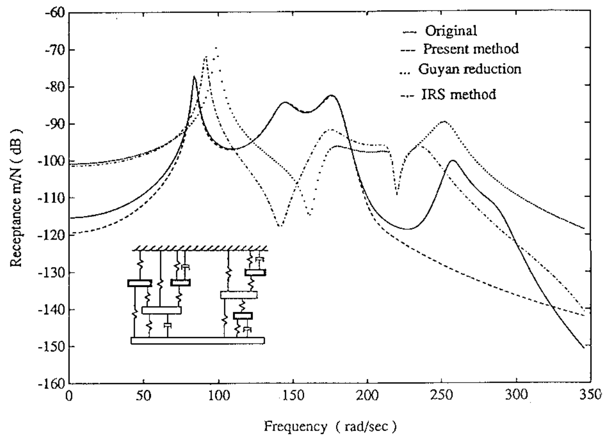

Fig. 5 Receptance FRF $h_{52}(\omega)$ obtained by selecting x_1, x_2, x_5, and x_6 as master DOFs for the 7-DOF system.

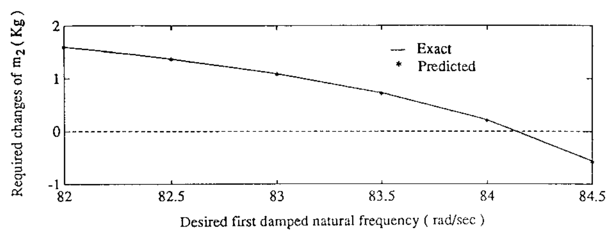

Fig. 6 Modification of mass m_2 required for desired first damped natural frequency obtained by selecting x_1, x_2, x_5, and x_6 as master DOFs for the 7-DOF system.

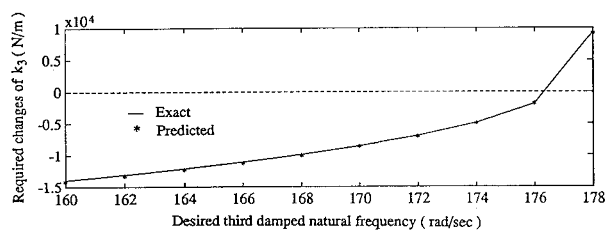

Fig. 7 Modification of stiffness k_3 required for desired third damped natural frequency obtained by selecting x_1, x_4, x_5, and x_6 as master DOFs for the 7-DOF system.

desired first damped natural frequency and the required changes of mass m_2 with master DOFs x_1, x_2, x_5, and x_6. The predicted results are fairly accurate. For instance, the first damped natural frequency of the original system is 84.17 rad/s. It is desired to shift the frequency to 82 rad/s by modifying the mass m_2. The exact solution for δm_2 is 1.594 kg, and the predicted δm_2 is 1.595 kg. The relative error is −0.09%. Figure 7 shows the relation between the desired third damped natural frequency and the required changes of stiffness k_3 between mass m_4 and mass m_5 with master DOFs x_1, x_4, x_5, and x_6. For example, the third damped natural frequency of the original system is 176.73 rad/s. It is desired to shift the frequency to 170 rad/s by modifying the stiffness k_3. The exact solution for δk_3 is −8546 N/m, and the predicted δk_3 is −8624 N/m. The relative error is 0.91%.

Example 3

An experimental example is used for this case. The test structure is a free-free acrylic beam. The dimensions and material properties are shown in Fig. 8. Since the acrylic beam is supported with soft and light elastic bands, the structure can be considered "freely" suspended in space. For the free-free acrylic beam structure, seven points with equal spacing are selected as the measurement locations (nodes) as shown in Fig. 8. Modal testing was performed on the selected locations to obtain the required FRFs. Only the responses in the x direction, as shown in Fig. 8, are measured. The identified modal parameters of the original beam structure are listed in Table 3. In this example, the sampling frequency ($\Delta\omega$) is 1.0 Hz

Table 3 Eigenvalues of the original and the modified free-free acrylic beam

	Eigenvalue $\lambda_r = -\sigma_r + i\omega_r$					
	Original		Modified by adding a mass block on the fourth node		Modified by adding a mass block on the second node	
Mode	σ_r, Hz	ω_r, Hz	σ_r, Hz	ω_r, Hz	σ_r, Hz	ω_r, Hz
1	1.88	54.75	1.78	50.68	1.84	54.02
2	4.70	152.46	4.67	152.06	3.74	150.20
3	7.92	299.73	7.59	275.62	8.07	283.87

Table 4 Comparison of actual and predicted results after mass modification for free-free acrylic beam

Master DOFs / Added mass, kg	δm_4 for first mode		δm_2 for second mode
	x_1–x_7	x_1–x_4	x_1–x_7
Actual	0.227	0.227	0.227
Predicted	0.226	0.226	0.226
Error, %	0.39	0.53	0.51

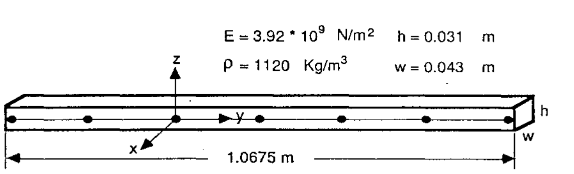

Fig. 8 Dimensions and material properties of the free-free acrylic beam.

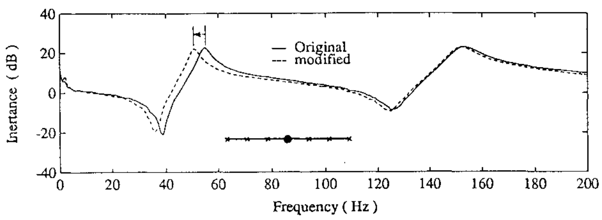

Fig. 9 Inertance FRFs $h_{11}(\omega)$ of the free-free acrylic beam before and after modification.

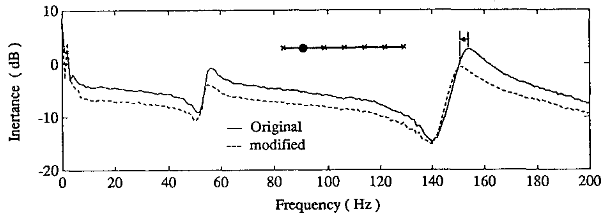

Fig. 10 Inertance FRFs $h_{22}(\omega)$ of the free-free acrylic beam before and after modification.

and nine frequency points, namely, 50.0, 60.0, 70.0, 140.0, 150.0, 160.0, 290.0, 300.0, and 310.0 Hz are used for the identification procedure of the present method.

The first case for structural modification is to shift the first damped natural frequency from 54.75 Hz to the desired frequency of 50.68 Hz by adding a mass block on the middle point (fourth node) of the beam. By using the present method, the predicted mass is 0.226 kg. Since the mass actually added on the structure is 0.227 kg, the relative error is 0.39%. The second case is to shift the second damped natural frequency from 152.46 Hz to the desired frequency of 150.20 Hz by adding a mass block on the second node of the beam. The result is listed in Table 4. Figures 9 and 10 show the inertance FRFs of the original and modified systems for these two cases, respectively. From Fig. 9, it is observed that since the middle point is a nodal point (zero displacement) of the second mode of the original beam structure, the second mode is almost not shifted after modification. The same holds true for the first mode as shown in Fig. 10. A critical case is also studied. When only the first, second, third, and fourth nodes are selected as the master DOFs, the present method can also accurately predict the changes of the mass at the middle point of the original beam structure. The relative error is 0.53% as listed in Table 4.

VI. Conclusions

A frequency-domain method based on the condensed normal FRFs extracted from the measured complex FRFs of real structures is developed for identifying the condensed mass, stiffness, and damping matrices. In this method, the estimation of the damping matrix is independent from that of the mass and the stiffness matrices, and the relative error is small. These identified condensed physical system matrices provide an accurate dynamic condensation of the original full system. It is emphasized that the method is practical because the structural modification for shifting the damped natural frequency to a desired value can be achieved directly from the experimental FRFs by applying the present method. There is no need to construct a finite element model.

Appendix: Summary of the Structural Modification

The equations of motion for a modified system without external excitation can be expressed as

$$(M + \Delta M)\ddot{x}^C(t) + (C + \Delta C)\dot{x}^C(t) + (K + \Delta K)x^C(t) = \mathbf{0} \quad \text{(A1)}$$

where ΔM, ΔC, and ΔK are the changes of mass, damping, and stiffness of the original system, respectively. By taking the Laplace transform of Eq. (A1) with zero initial conditions, we have

$$(\lambda^2 M + \lambda C + K)x^C(\lambda) = -(\lambda^2 \Delta M + \lambda \Delta C + \Delta K)x^C(\lambda) \quad \text{(A2)}$$

where $\lambda = -\sigma + i\omega$. Since the left-hand side of Eq. (A2) is the inverse of the transfer function matrix of the original system, Eq. (A2) can be rewritten as

$$x^C(\lambda) = -H(\lambda)(\lambda^2 \Delta M + \lambda \Delta C + \Delta K)x^C(\lambda) \quad \text{(A3)}$$

It is assumed that ΔM and ΔK have following simple forms. First, consider modifying a lumped mass at node Q with δm_Q, the matrix ΔM_Q can be expressed as

$$\Delta M_Q = \delta m_Q \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & \cdots & & \\ & & & 1 & \\ & & & & \cdots \\ & & & & & 0 \end{bmatrix} \quad \text{(A4)}$$

Next, for a spring with a modification of stiffness δk_p between node I and node J, the matrix ΔK_p can be represented as

$$\Delta K_p = \delta k_p \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & a_{II} & \cdots & 0 & \cdots & a_{IJ} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & a_{JI} & \cdots & 0 & \cdots & a_{JJ} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}_p \quad \text{(A5)}$$

Detailed discussion of the ΔM and ΔK were presented by Tsuei and Yee.[11] The damped natural frequency can be shifted by changing the mass and stiffness parameters. If only a simple stiffness P is modified, Eq. (A3) is reduced to

$$-\begin{bmatrix} h_{II}(\lambda) & h_{IJ}(\lambda) \\ h_{JI}(\lambda) & h_{jj}(\lambda) \end{bmatrix} \begin{bmatrix} a_{II} & a_{IJ} \\ a_{JI} & a_{JJ} \end{bmatrix}_p \begin{bmatrix} x_I \\ x_J \end{bmatrix}^C = \frac{1}{\delta k_p} \begin{bmatrix} x_I \\ x_j \end{bmatrix}^C \tag{A6}$$

It can be considered as

$$J(\lambda)z = \gamma z \tag{A7}$$

The λ and γ are the unknown parameters in Eq. (A7). If the variable λ Eq. (A7) is known, the equation becomes an eigenvalue problem. When a damped natural frequency of the system is shifted from the original value ω_{org} to shifted value ω_s, the imaginary part of λ becomes ω_s, i.e., $\lambda = -\sigma + i\omega_s$. The matrix $J(\lambda)$ cannot be computed because the real part of λ is unknown, but λ and γ can be determined using an iterative procedure.

During the iteration, an initial value of σ, denoted as σ_a, is assumed for the real part of λ. The corresponding value of λ is defined as λ_a, i.e., $\lambda_a = -\sigma_a + i\omega_s$. If a value of λ_a is assumed and the corresponding γ_a value is determined from Eq. (A7) but the value of γ_a is not a real number (since δk_p is real and is the inverse of γ, so γ must be a real number), the values of λ_a and γ_a are not the possible solutions. A different value of λ_a has to be assumed, and the calculation process repeated.

Next, if a mass parameter m_Q is changed, Eq. (A3) is reduced to

$$\boldsymbol{x}_Q^C = -\lambda^2 \delta m_Q h_{QQ}(\lambda) \boldsymbol{x}_Q^C \tag{A8}$$

For a particular λ_a, it can be written as

$$\delta m_Q = -1/\lambda_a^2 h_{QQ}(\lambda_a) \tag{A9}$$

Similar to the previous procedures of the stiffness modification, a desired damped natural frequency ω_s is specified and a value of σ_a is assumed. The desired system eigenvalue λ_s is obtained when the imaginary part of $\lambda_a^2 h_{QQ}(\lambda_a)$ is zero. The mass change δm_Q at node Q is calculated from Eq. (A9).

Equation (A7) may provide more than one realistic δk_p, but only one δk_p is the correct modification for shifting a particular system mode from original damped natural frequency ω_{org} to a desired value ω_s. The nonunique solutions indicate that the frequency of different modes can be shifted. For instance, there are two possible solutions: one is positive δk_p, and one is negative δk_p. The positive δk_p indicates that the frequency of a lower mode, rth mode, is shifted up from ω_r to ω_s, where $\omega_r < \omega_s$. The negative δk_p indicates that the frequency of a higher mode, uth mode, is shifted down from ω_u to ω_s, where $\omega_u > \omega_s$. The user has to select the appropriate δk_p to achieve the desired modification. It is noted that we would like to modify the structure with smallest changes, i.e., to choose the largest eigenvalue from Eq. (A7).

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References

[1]Guyan, R. J., "Reduction of Stiffness and Mass Matrices," *AIAA Journal*, Vol. 3, No. 2, 1965, p. 380.

[2]O'Callahan, J., "A Procedure for An Improved Reduced System (IRS) Model," *Proceedings of 7th International Modal Analysis Conference* (Las Vegas, NV), Society for Experimental Mechanics, Inc., Bethel, CT, 1989, pp. 17–21.

[3]Paz, M., "Dynamic Condensation," *AIAA Journal*, Vol. 22, No. 5, 1984, pp. 724–727.

[4]Ibrahim, S. R., "Dynamic Modeling of Structures from Measured Complex Modes," *AIAA Journal*, Vol. 21, No. 6, 1983, pp. 898–901.

[5]Fritzen, C.-P., "Identification of Mass, Damping, and Stiffness Matrices of Mechanical Systems," *Journal of Vibration, Acoustics, Stress, and Reliability in Design*, Vol. 108, Jan. 1986, pp. 9–16.

[6]Wang, J.-H., "Mechanical Parameters Identification, with Special Consideration of Noise Effects," *Journal of Sound and Vibration*, Vol. 125, No. 1, 1988, pp. 151–167.

[7]Minas, C., and Inman, D. J., "Identification of a Nonproportional Damping Matrix from Incomplete Modal Information," *Journal of Vibration and Acoustics*, Vol. 113, April 1991, pp. 219–224.

[8]Roemer, M. J., and Mook, D. J., "Mass, Stiffness, and Damping Matrix Identification: An Integrated Approach," *Journal of Vibration and Acoustics*, Vol. 114, July 1992, pp. 358–363.

[9]Chen, S. Y., Ju, M. S., and Tsuei, Y. G., "Estimation of Mass, Stiffness, and Damping Matrices from Frequency Response Functions," *Journal of Vibration and Acoustics*, Oct. 1995, pp. 1–5.

[10]Yee, E. K. L., and Tsuei, Y. G., "Method for Shifting Natural Frequencies of Damped Mechanical Systems," *AIAA Journal*, Vol. 29, No. 11, 1991, pp. 1973–1977.

[11]Tsuei, Y. G., and Yee, E. K. L., "A Method for Modifying Dynamic Properties of Undamped Mechanical Systems," *ASME Journal of Dynamic Systems, Measurement, and Control*, Vol. 111, Sept. 1989, pp. 403–408.